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Finding

A special function is a real or complex valued function of one or more real or complex variables which is specified so completely that its numerical values could in principle be tabulated. Besides elementary functions such as x^n, e^x, $\ell_n(x)$, and $\sin x$, "higher" functions, both transcendental (such as Bessel functions) and algebraic (such as various polynomials) come under the category of special functions. In fact, special functions are solutions of a wide class of mathematically and physically relevant functional equations. As far as the origin of special functions is concerned the special function of mathematical physics arises in the solution of partial differential equations governing the behavior of certain physical quantities.

Special functions and their applications are now awe-inspiring in their scope, variety and depth. Not only in their rapid growth in pure mathematics and their applications to the traditional fields of physics, engineering and statistics but in new fields of applications like behavioral science, optimization, biology, environmental science and economics etc they are emerging.

The present thesis comprises of **Ten Chapters**. A brief summary of the problems is presented at the beginning of each chapter and then each chapter is divided into a number of sections. Equations in every section are numbered separately. For example, the small bracket (a.b.c) specified the results, in which the last figure denotes the equation number, the middle one denotes the section and the first indicates the chapter to which it belongs. Sections, Articles, Definitions and Equations have been numbered chapter wise.

The aim of the **Chapter 1** is to introduce several classes of special functions which occur rather more frequently in the study of summations and transformations needed for the presentation of subsequent chapters. We discuss different forms of Gamma function; Pochhammer symbol; ordinary hypergeometric function of one variable and its convergence conditions; General hypergeometric functions of Kampé de Fériet; Multiple hypergeometric functions of Kampé de Fériet; General double hypergeometric function of Srivastava-Daoust and many other functions.

It provides a systematic introduction to most of the important special functions that commonly arise in practice and explores many of their silent properties. This chapter is also intended to make the thesis as much self contained as possible.

In **Chapter 2**, we obtain successive differentiation and successive integration of some typical mathematical functions like: $-\frac{4}{z}\ell n\left(\frac{1+\sqrt{(1-z)}}{2}\right)$; $(z)^{\frac{1}{2}}\sin^{-1}\sqrt{(z)}+\sqrt{(1-z)}$; $(z)^{-\frac{1}{2}}\sin^{-1}\sqrt{(z)}+\sqrt{(1-z)}$; $\frac{4}{z}\left[1-\sqrt{(1-z)}+\ell n\left(\frac{1+\sqrt{(1-z)}}{2}\right)\right]$ and $\frac{4}{z^2}\left[2\sqrt{(1-z)}-2+z-2z\ell n\left(\frac{1+\sqrt{(1-z)}}{2}\right)\right]$ by using the hypergeometric functions approach; as the successive differentiation and successive integration of these functions can not be performed by any other mathematical technique. Further, by changing the independent and dependent variables in the suitable ordinary differential equations of second and third order and comparing the results of ordinary differential equations with standard ordinary hypergeometric differential equations of Gauss and Clausen, we obtain the hypergeometric form of the functions: $\frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}}$, $[\sin^{-1}(x)]^2$ *and* $\sin^{-1}(x)$.

In **Chapter 3**, we obtain analytical solutions of an important improper integral $\left(\int_0^\infty \frac{x^{s-1}}{(a-x)}dx; a>0\right)$ of complex analysis, by using series manipulation and Dixon's summation theorem for Clausen series ${}_3F_2(1)$. Further, we apply it in evaluation of other improper integrals based on Mellin transforms. We also acquire analytical solutions of some real definite integrals (whose integrands are the rational functions of sine and cosine functions) by making use of binomial theorem, decomposition of single and double infinite series, properties of definite integral and Pfaff-Kümmer linear transformation. In addition other typical definite integrals based on beta function are also obtained by making use of binomial theorem, Kümmer's first summation theorem and Legendre's duplication formula.

In **Chapter 4**, using series rearrangement technique, we derive a Whipple's type new quadratic transformation for Clausen function ${}_3F_2[a,\, b,\, c;\, 2+a-b,\, 2+a-c;\, z]$ and a quadratic transformation for Gauss function ${}_2F_1[a,\, b;\, 2+a-b;\, z]$. We also obtain a family of hypergeometric summation theorems for Clausen function having the arguments ± 1. Further, we obtain three general double-series identities involving the bounded sequences of arbitrary complex numbers employing the Bailey transformations for terminating ${}_4F_3$ hypergeometric series with unit argument. As applications of these three general double-series identities, we establish some transformations formulas (believed to be new) for Kampé de Fériet's double hypergeometric functions with equal arguments. Some Clausen reduction formula, Karlsson reduction formula, Orr reduction formula and other new transformations are also deduced as a special cases of our main transformations.

In **Chapter 5**, we obtain some Laurent's type hypergeometric generating relations for certain mixed special functions related to the Bessel functions using series rearrangement technique. Further, some Maclaurin's type generating relations for Appell's functions, Lauricella's functions and multiple hypergeometric functions of Kampé de Fériet are also given. In addition two general double-series identities involving bounded sequences of arbitrary complex numbers employing the finite summation theorems of Gessel-Stanton and G. Andrews for terminating ${}_3F_2$ hypergeometric series with arguments $3/4$ and $4/3$, respectively. Using these double-series identities, we establish two reduction formulas for the Srivastava-Daoust double hypergeometric function with arguments $z, 3z/4$ and $z, -4z/3$ expressed in terms of two generalized hypergeometric function of arguments proportional to z^3 and $-z^3$ respectively.

In **(Chapter 6 and Chapter 7)**, we obtain some generalizations and unifications (with suitable convergence conditions) of the Gauss, Kümmer and Rakha-Rathie-Chopra quadratic transformations by using new sequence introduced by Choi *et al* and series rearrangement technique. Next, by differentiating both sides of suitable formula presented here with respect to a parameter, we obtain another quadratic transformation as well as some numerical verifications.

In **Chapter 8**, we obtain four general double series identities having the bounded sequences by using one hypergeometric summation theorem for terminating ${}_3F_2\,(1)$, three hypergeometric transformations for terminating ${}_4F_3\,(1)$ and series rearrangement technique. Further, we establish some transformations (not recorded earlier in the literature) for Kampé de Fériet's double hypergeometric function with equal arguments into generalized hypergeometric function with argument z and three Srivastava-Daoust double hypergeometric functions having the arguments $z, -\frac{z^2}{4}$; $4z, -4z^2$ and $z, \frac{z^2}{4}$. We have also verified these hypergeometric transformations (in-fact, which are the generalizations of the reduction formula of Clausen) using Mathematics software.

In **Chapter 9**, three general double-series identities involving the bounded sequences of arbitrary complex numbers employing the Whipple transformations for terminating ${}_4F_3$ and ${}_5F_4$ hypergeometric series with unit argument are obtained. As applications of these three general double-series identities, we establish three transformation formulas (believed to be new) for Srivastava-Daoust double hypergeometric function with arguments $-4z^2$, $4z$; $-\frac{z^2}{4}, z$ and $-z$, z expressed in terms of Kampé de Fériet's double hypergeometric function of equal arguments. Some Bailey's quadratic transformations, Clausen's reduction formula, Gauss quadratic transformation, Karlsson reduction formula, Orr reduction formula, Whipple quadratic transformation and other new transformations are also deduced as special cases of our main transformations.

In **Chapter 10**, we derive the analytical expressions for the exact length of the arc of semi loops (not previously recorded in literature) of the petals-shaped curves for example: $r = a\sin(2p\theta)$, $r = a\sin((2p+1)\theta)$, $r = a\cos(2p\theta)$ and $r = a\cos[(2p+1)\theta]$ in terms of Kampé de Fériet double hypergeometric function and Clausen function ${}_3F_2$. We have also obtained the exact expressions for the curved surface area of the three dimensional figures when axes of revolution are initial line and a line perpendicular to initial line. Further, as an application we have also obtained the exact circumference of one complete loop and total length of all loops of the above curves. All the obtained results are believed to be new. The results derived in this chapter are fascinating and may be useful in the applicable sciences. We have verified the analytical expressions numerically using *Mathematica Program.*