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Name of the Scholar	: Bilal Ahmad Bhat
Name of the Supervisor	: Prof. Mohd. Idris Qureshi
Name of the Department/Centre	: Applied Sciences and Humanities, Faculty of Engineering and Technology
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Finding

The theory of special functions is a branch of Mathematics, whose roots are deeply penetrated in analysis, the theory of functions of a complex variable, the theory of group representations, and the theoretical and mathematical physics. Special functions have been around for centuries. Once a function is added to the list of 'special functions', its use to solve problems becomes acceptable. So, in this way, special functions constitute a common currency of mathematics. We can say that hypergeometric function lies at the heart of the theory of special functions. All special functions can be expressed in terms of hypergeometric functions. Hypergeometric functions opened many new areas of mathematical approach for investigation in physics, engineering and biology. Thus, study of hypergeometric functions is of fundamental importance in the field of mathematical sciences.

In view of this inherent relationship the hypergeometric functions occupied a place of pride in subsequent research and helped in unifying the scattered mass of results and takes a prominent position in mathematics, both pure and applied, and in many branches of science. In a nutshell, special functions are solutions of a wide class of mathematically and physically relevant functional equations. These functions arise as the solutions of ordinary or partial differential equations, which are used to express the problems mathematically in different branches of sciences, for example, heat equation in physics, spectral problems in quantum mechanics like establishing eigenvalues of Schrödinger operator applications, conduction, communication systems, electro optics, non-linear wave propagation, electromagnetic theory, statics, dynamics, quantum mechanics, fluid dynamics, statistical communication theory, fiber optics, electro-chemistry, diffusion, rheology, quantitative biology, scattering theory, transport theory, heat conduction in solids, vibration phenomena, approximation theory, probability theory.

In the present work, an attempt has been made to study the **ANALYTICAL AND THEORETICAL EFFICACY OF HYPERGEOMETRIC METHODOLOGY**. This thesis comprises of **Ten chapters**. A brief summary of the problems is set up at the beginning of each chapter and then each chapter is categorized into a number of sections. Equations in every section are numbered separately. For example the small bracket (a.b.c) specified the results, in which the last figure denotes the equation number, the middle one denotes the section and the first indicates the chapter to which it belongs. Sections, Articles, Definitions and Equations have been numbered chapter wise.

Chapter-1 is devoted to the brief survey of the literature of the subject. It aims at introducing several classes of special functions, which occur rather more frequently in the study of summations and transformations needed for the presentation of subsequent chapters. We discuss Gamma and Beta functions, and their important properties. We then move towards the hypergeometric functions and their generalizations (as Appell function, double hypergeometric function of Kampé de Fériet, double hypergeometric function of Srivastava-Daoust) in one and two variables. We also present the definitions of some hypergeomtric theorems and polynomials which include Bedient Polynomials, Zero-balanced series, Karlsson-Srivastava terminating series and also some integrals of Srinivasa Ramanujan and other definite integrals. Finally, we set the various properties of Orthogonal polynomials and discuss the perimeter and curved surface areas of Limacon and Cardioid.

The aim of the **Chapter 2** is to to establish four general double-series identities, which involve some suitably bounded sequences of complex numbers, by using zero-balanced terminating hypergeometric summation theorems for the generalized hypergeometric series ${}_{r+1}F_r\,(1)$ in conjunction with the series rearrangement technique. The sum (or difference) of two general double hypergeometric functions of the Kampe de Feriet type are then obtained in terms of a generalized hypergeometric function under appropriate convergence conditions. A closed form of the following Clausen hypergeometric function: ${}_3F_2\left(\frac{-27z}{4(1-z)^3}\right)$ and a reduction formula for the Srivastava-Daoust double hypergeometric function with the arguments $(z, \frac{-z}{4})$ are also derived.

In **Chapter 3**, we derive eighteen addition theorems (believed to be new and not found in the literature) for the product of two Chebyshev polynomials of first, second, third and fourth kinds, together with two Gegenbauer, two Legendre, two Fibonacci and two Lucas polynomials denoted by $T_n(x)$, $U_n(x)$, $V_n(x)$, $W_n(x)$, $C_n^{\nu}(x)$, $P_n(x)$, $F_n(x)$ and $L_n(x)$ respectively, using suitable generating relations, decomposition of infinite series, Cauchy double-series identity and series rearrangement technique. We have also verified these eighteen addition theorems numerically, using *Mathematica* software and Tables for $T_n(\pm\frac{1}{2})$, $U_n(\pm\frac{1}{2})$, $V_n(\pm\frac{1}{2})$ and $W_n(\pm\frac{1}{2})$, $C_n^3(\pm\frac{1}{2})$, $P_n(\pm\frac{1}{2})$, $F_n(\pm i)$ and $L_n(\pm i)$.

In **Chapter 4**, we obtain some hypergeometric generating relations involving Kampé de Fériet's double hypergeometric functions by using series rearrangement technique, a reduction formula and Whipple's quadratic transformation. Some special cases involving Appell's functions of the second and third kinds, Rainville polynomial and two Bedient's polynomials, are also obtained. Moreover, we provide the sum of the squares and cubes of the coefficients of non-negative integral powers of z in the binomial expansion of $(1-z)^{-\lambda}$, by using Gauss and Dixon summation theorems. Further, we discuss some interesting particular cases of the above-referred binomial expansion.

In **Chapter 5**, we derive four general double-series identities having the bounded sequences, by employing Karlsson and Karlsson-Srivastava terminating summation theorems ${}_3F_2\,(4)$, ${}_3F_2\left(\frac{1}{4}\right)$ along with series rearrangement technique. Further, four reduction formulas for Srivastava-Daoust double hypergeometric functions with arguments $(z, -4z)$ and $\left(z, \frac{-z}{4}\right)$ are also obtained, in terms of single generalized hypergeometric functions with arguments $\pm(27)^{(D-E-1)}z^3$ and $\mp(27)^{(D-E-1)}z^3$ respectively, associated with suitable convergence conditions.

In **Chapter 6**, we provide alternative proofs of some favorite integrals of Srinivasa Ramanujan, based on Mellin transforms and other integrals (with suitable convergence conditions) by using the beta function, recurrence relation and suitable substitutions. Moreover, we provide the analytical solutions of two definite integrals $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos\theta \pm \sin\theta)^{\alpha}d\theta$ (with suitable convergence conditions) by using the approaches of gamma and hypergeometric functions. Further, we also obtain a summation formula for Kampé de Fériet's double hypergeometric function having the arguments ± 1.

In **Chapter 7**, by using the hypergeometric approach we evaluate four definite integrals having the integrand (in the form of real powers of $\tan\theta$), in order to get various summation formulas for Srivastava-Daoust double hypergeometric functions containing the arguments ± 1.

In **Chapter 8**, we acquire some addition theorems (believed to be new and not recorded in the literature) for the product of two generalized Hermite polynomials, several classical Hermite polynomials and Laguerre polynomials, using suitable generating relations, decomposition of infinite series, Cauchy double and multiple-series identities and series rearrangement technique. Further, we discuss some special cases related to the generalized and classical Hermite polynomials.

In **Chapter 9**, we provide the proofs of some typical definite integrals like Nicholson integral, Lavoie-Trottier integral, MacRobert's integral, Nielsen integral, Edwards integral, Saxena integrals, Bierens de Haan D. integral and Fikhtengol'ts G.M. integral (whose solutions are not available in the related research papers of the concerned authors as well as in any mathematical tables), by using decomposition technique, Srivastava identity, Euler's linear transformation, Kummer's first transformation, a transformation formula of Bailey, Karlsson's summation theorem, hypergeometric forms of the error-function, arcsine function, Murphy formula and Cahen-Mellin integral.

In **Chapter 10**, we aim to obtain the analytical expressions for the exact length (not previously recorded in the literature) of an arbitrary arc of Limacon, in terms of Kampé de Fériet double hypergeometric function and Clausen function ${}_3F_2$. Further, we have also deduced the perimeters of Limacon, Cardioid and a new summation theorem for Clausen function ${}_3F_2$. We have also given the exact expressions for the curved surface areas of revolution (not previously recorded in the literature) obtained by revolving the arbitrary arc of Limacons and Cardioids when axes of revolution are the initial line and a line perpendicular to the initial line (passing through the pole). Some interesting case studies of curved surface areas of revolution are also discussed.